GENERALIZED SERRE DUALITY

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ABSTRACT. We introduce a notion of generalized Serre duality on a Hom-finite Krull-Schmidt triangulated category $\mathcal{T}$. This duality induces the generalized Serre functor on $\mathcal{T}$, which is a linear triangle equivalence between two thick triangulated subcategories of $\mathcal{T}$. Moreover, the domain of the generalized Serre functor is the smallest additive subcategory of $\mathcal{T}$ containing all the indecomposable objects which appear as the third term of an Auslander-Reiten triangle in $\mathcal{T}$; dually, the range of the generalized Serre functor is the smallest additive subcategory of $\mathcal{T}$ containing all the indecomposable objects which appear as the first term of an Auslander-Reiten triangle in $\mathcal{T}$.

We compute explicitly the generalized Serre duality on the bounded derived categories of artin algebras and of certain noncommutative projective schemes in the sense of Artin and Zhang. We obtain a characterization of Gorenstein algebras: an artin algebra A is Gorenstein if and only if the bounded homotopy category of finitely generated projective A-modules has Serre duality in the sense of Bondal and Kapranov.

1. INTRODUCTION

Throughout R will be a commutative artinian ring. Let $\mathcal{T}$ be an R-linear triangulated category which is *Hom-finite* and *Krull-Schmidt*. Here, an R-linear category is Hom-finite provided that all its Hom spaces are finite generated R-modules; an additive category is Krull-Schmidt provided that each object is a finite sum of indecomposable objects with local endomorphism rings.

The Auslander-Reiten theory for a Hom-finite Krull-Schmidt triangulated category $\mathcal{T}$ was initiated by Happel ([12, 13]) which now plays an important role in the representation theory of artin algebras. The central notion is *Auslander-Reiten triangle*, which by definition is a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \to X[1]$ with X, Z indecomposable such that u (*resp.* v) is left (*resp.* right) almost-split. Here [1] denotes the translation functor on $\mathcal{T}$. For an indecomposable object X there exists at most one, up to isomorphism, Auslander-Reiten triangle which contains X as the first term; dually, for an indecomposable object Z there exists at most one Auslander-Reiten triangle which contains Z as the third term. The triangulated category $\mathcal{T}$ is said to *have enough Auslander-Reiten triangles* provided that for each indecomposable object X there exist two Auslander-Reiten triangles which contain X as the first term and the third term, respectively. A fundamental result due to Reiten and Van den Bergh states that $\mathcal{T}$ has enough Auslander-Reiten triangles if and only if it has Serre

duality ([26, Theorem I.2.4]). Here the Serre duality is in the sense of Bondal and Kapranov ([7]).

Observe that in general a Hom-finite Krull-Schmidt triangulated category $\mathcal{T}$ will not have enough Auslander-Reiten triangles. For example, Happel proved that the bounded derived category of an artin algebra has enough Auslander-Reiten triangles if and only if the algebra has finite global dimension ([13]). In this situation, the category $\mathcal{T}$ will not have Serre duality.

In the present paper, we introduce a notion of *generalized Serre duality* on a Hom-finite Krull-Schmidt triangulated category $\mathcal{T}$. This generalized duality induces the *generalized Serre functor* of $\mathcal{T}$, which is a linear triangle equivalence between certain *thick* triangulated subcategories of $\mathcal{T}$. Here, thick subcategories mean subcategories which are closed under direct summands. We put Reiten-Van den Bergh's theorem in this general setting. It turns out that this generalized Serre duality is closely related the Auslander-Reiten triangles in $\mathcal{T}$.

The notion of generalized Serre duality applies to an arbitrary Hom-finite R-linear category. We need some notation. Let $\mathcal{C}$ be a Hom-finite R-linear category. For two objects X and Y, we write (X, Y) for the Hom space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Denote by $(X, -)$ (*resp.* $(-, X)$) the representable functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ (*resp.* $\mathrm{Hom}_{\mathcal{C}}(-, X)$) from $\mathcal{C}$ to the category R-mod of finitely generated R-modules. Denote by D the Matlis duality on R-mod. Recall that $D = \mathrm{Hom}_R(-, E)$ where E is the minimal injective cogenerator for R; see [5, Chapter II.3]. By $D(X, -)$ (*resp.* $D(-, X)$) we mean the composite functor $D\mathrm{Hom}_{\mathcal{C}}(X, -)$ (*resp.* $D\mathrm{Hom}_{\mathcal{C}}(-, X)$).

We define two full subcategories of $\mathcal{C}$ as follows

$$\mathcal{C}_r = \{X \in \mathcal{C} \mid D(X, -) \text{ is representable}\} \text{ and } \mathcal{C}_l = \{X \in \mathcal{C} \mid D(-, X) \text{ is representable}\}.$$

It turns out that there exists a unique functor $S \colon \mathcal{C}_r \to \mathcal{C}_l$ such that there is a R-linear isomorphism

$$\phi_{X,Y} \colon D(X, Y) \simeq (Y, S(X))$$

for each $X \in \mathcal{C}_r$ and $Y \in \mathcal{C}$; the isomorphism $\phi_{X,Y}$ is required to be natural both in X and Y. In fact, the functor S is an R-linear equivalence of categories; see Appendix A.1. We will call the functor S the *generalized Serre functor* on $\mathcal{C}$, and we refer to $\mathcal{C}_r$ (*resp.* $\mathcal{C}_l$) as the *domain* (*resp.* the *range*) of the generalized Serre functor. We will call this set of data the *generalized Serre duality* on $\mathcal{C}$. Observe that the category $\mathcal{C}$ has Serre duality in the sense of [7] if and only if $\mathcal{C}_r = \mathcal{C} = \mathcal{C}_l$. For details, see Appendix.

Here is our main theorem.

Main Theorem. *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category. Denote by $S \colon \mathcal{T}_r \to \mathcal{T}_l$ its generalized Serre functor. Then both $\mathcal{T}_r$ and $\mathcal{T}_l$ are thick triangulated subcategories of $\mathcal{T}$. Moreover, we have*

(1) *there is a natural isomorphism $\eta_X \colon S(X[1]) \to S(X)[1]$ for each $X \in \mathcal{T}_r$ such that the pair (S, η) is a triangle equivalence between $\mathcal{T}_r$ and $\mathcal{T}_l$;*

(2) *an indecomposable object X in $\mathcal{T}$ belongs to $\mathcal{T}_l$ (resp. $\mathcal{T}_r$) if and only if there is an Auslander-Reiten triangle in $\mathcal{T}$ containing X as the first term (resp. the third term).*

Let us remark that the statement (1) is a generalization of a result due to Bondal and Kapranov ([7]; also see [6, 16]). The main theorem is a combination of Propositions 2.5, 2.7, 2.8 and A.1.

The paper is organized as follows. In section 2, we divide the proof of the main theorem into proving several propositions. Section 3 is devoted to computing the generalized Serre duality for the bounded derived categories of certain abelian categories explicitly; see Theorem 3.5 and Theorem 3.10. The computational results are based on results by Happel ([13]), and by de Naeghel and Van den Bergh ([24]). We obtain, as a byproduct, a seemingly new characterization of Gorenstein algebras: an artin algebra A is Gorenstein if and only if the bounded homotopy category of finitely generated projective A-modules has Serre duality; see Corollary 3.9. The appendix explains the basic notions and results on generalized Serre duality on a Hom-finite linear category.

For triangulated categories, we refer to [12, 15, 23]. For the notion of Auslander-Reiten triangles, we refer to [12, 13].

2. Proof of Main Theorem

In this section, we divide the proof of the main theorem into proving several propositions. We make preparation on a study of coherent functors on triangulated categories. We collect the relevant results which seems to be scattered in the literature.

Let $\mathcal{C}$ be an additive category. Denote by $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the large category of contravariant additive functors from $\mathcal{C}$ to the category Ab of abelian groups. Note that although $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is usually not a category, but it still makes sense to mention the exact sequences of functors, in particular, the notions of kernel, cokernel and extension in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ make sense. Hence sometimes we even pretend that $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian category.

Recall that the Yoneda embedding functor $\mathbf{p}\colon \mathcal{C} \to (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is defined by $\mathbf{p}(C) = (-, C)$, where $(-, C)$ denotes the representable functor $\mathrm{Hom}_{\mathcal{C}}(-, C)$. Then Yoneda Lemma implies that $\mathbf{p}$ is fully faithful; moreover, representable functors are projective objects in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Recall that a functor $F \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is said to be *coherent* provided that there exists an exact sequence of functors

$$(-, C_0) \longrightarrow (-, C_1) \longrightarrow F \longrightarrow 0,$$

where $C_0, C_1 \in \mathcal{C}$. Note that such a sequence can be viewed as a projective presentation of F in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Denote by $\widehat{\mathcal{C}}$ the full subcategory of $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ consisting of coherent functors on $\mathcal{C}$. Note that the category $\widehat{\mathcal{C}}$ of coherent functors is an additive category with small Hom sets.

Recall that a *pseudokernel* of a morphism $f\colon C_0 \to C_1$ in $\mathcal{C}$ is a morphism $k\colon K \to C_0$ such that $f \circ k = 0$ and that each morphism $k'\colon K \to C_0$ with $f \circ k' = 0$ factors through k, in other words, the morphism k makes the sequence $(-, K) \overset{(-,k)}{\to} (-, C_0) \overset{(-,f)}{\to} (-, C_1)$ of functors exact. We say that the category $\mathcal{C}$ *has pseudokernels* provided that each morphism has a pseudokernel. Dually, one defines *pseudocokernels*.

Let us recall a basic result on coherent functors ([2, 3]; also see [19]).

Lemma 2.1. *Let $\mathcal{C}$ be an additive category. Then the category $\widehat{\mathcal{C}}$ of coherent functors is closed under cokernels and extensions in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. In particular, $\widehat{\mathcal{C}}$ is closed under*

taking direct summands. Moreover, $\widehat{\mathcal{C}}$ is an abelian subcategory of $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ if and only if $\mathcal{C}$ has pseudokernels. $\square$

The following is well known, as pointed out in [19, Example 4.1(2)].

Lemma 2.2. *Let $\mathcal{T}$ be a triangulated category. Then $\mathcal{T}$ has pseudokernels and pseudocokernels.*

Proof. Given a morphism $v : Y \to Z$ in $\mathcal{T}$, we take a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

Since for any object $C \in \mathcal{T}$, the functors $(C, -)$ and $(-, C)$ are cohomological (see [15, p.23]), it is easily verified that u and w are a pseudokernel and a pseudocokernel of v, respectively. $\square$

In what follows $\mathcal{T}$ will be a triangulated category. We will introduce a *two-sided resolution* for any coherent functor on $\mathcal{T}$. Given a functor $F \in \widehat{\mathcal{T}}$, we have a presentation

$$(-, Y) \xrightarrow{(-,v)} (-, Z) \longrightarrow F \longrightarrow 0.$$

Take a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$. So we have a long exact sequence of functors

$$\cdots \longrightarrow (-, Z[-1]) \xrightarrow{(-,w[-1])} (-, X) \xrightarrow{(-,u)} (-, Y) \xrightarrow{(-,v)} (-, Z) \xrightarrow{(-,w)} (-, X[1]) \longrightarrow \cdots$$

Note that $F \simeq \mathrm{Coker}(-, v)$. Hence we have the following two exact sequences

$$\cdots \longrightarrow (-, Z[-1]) \xrightarrow{(-,w[-1])} (-, X) \xrightarrow{(-,u)} (-, Y) \xrightarrow{(-,v)} (-, Z) \longrightarrow F \longrightarrow 0,$$

and

$$0 \longrightarrow F \longrightarrow (-, X[1]) \xrightarrow{(-,u[1])} (-, Y[1]) \xrightarrow{(-,v[1])} (-, Z[1]) \xrightarrow{(-,w[1])} (-, X[2]) \longrightarrow \cdots$$

Denote by $\mathrm{Coho}(\mathcal{T})$ the full subcategory of $(\mathcal{T}^{\mathrm{op}}, \mathrm{Ab})$ consisting of cohomological functors. By Lemmas 2.1 and 2.2, we deduce that the category $\widehat{\mathcal{T}}$ is an abelian category which has enough projective objects. In particular, we may define the extension groups $\mathrm{Ext}^i(F, G)$ for any coherent functors F and G, $i \geq 1$.

The next lemma is also known; compare [23, p.258].

Lemma 2.3. *Let $H \in \widehat{\mathcal{T}}$. Then we have $H \in \mathrm{Coho}(\mathcal{T})$ if and only if $\mathrm{Ext}^i(F, H) = 0$ for all $i \geq 1$ and $F \in \widehat{\mathcal{T}}$.*

Proof. Assume that $F \in \widehat{\mathcal{T}}$. Note that we just obtained a projective resolution for F. We take the resolution to compute $\mathrm{Ext}^i(F, H)$. Apply Yoneda Lemma $((-, C), H) \simeq H(C)$. We obtain that $\mathrm{Ext}^i(F, H)$ is just the i-th cohomology group of the following complex

$$0 \longrightarrow H(Z) \longrightarrow H(Y) \longrightarrow H(X) \longrightarrow H(Z[-1]) \longrightarrow \cdots$$

Here we use the notation as in the above two-sided resolution. Now if H is cohomological, the above complex is exact at all the positions other than the zeroth one (where $H(Z)$ sits). Hence $\mathrm{Ext}^i(F, H) = 0$ for all $i \geq 1$.

On the other hand, assume that $\mathrm{Ext}^i(F, H) = 0$ for all $i \geq 1$ and $F \in \widehat{\mathcal{T}}$. To see that H is cohomological, take a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$. Take

$F = \text{Cok}(-, v)$; it is a coherent functor. As in the above discussion, the assumption that $\text{Ext}^i(F, H) = 0$ will imply that the complex

$$0 \longrightarrow H(Z) \longrightarrow H(Y) \longrightarrow H(X) \longrightarrow H(Z[-1]) \longrightarrow \cdots$$

is exact at all the positions other than the zeroth one. This implies that H is cohomological. $\qquad\square$

Denote by $\text{add}(\mathbf{p}(\mathcal{T}))$ the full subcategory of $(\mathcal{T}^{\text{op}}, \text{Ab})$ consisting of direct summands of representable functors. Observe that $\text{add}(\mathbf{p}(\mathcal{T})) \subseteq \text{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}}$.

The following result is due to Freyd and Verdier independently ([11, 29]). We refer to [23, p.169] and [19, 4.2] for a modern treatment. The result we present is in a slightly different form. Recall that an abelian category is *Frobenius* provided that it has enough projective and enough injective objects, and the class of projective objects coincides with the class of injective objects; see [12, Chapter I.2].

Proposition 2.4. *Let $\mathcal{T}$ be a triangulated category. Then the category $\widehat{\mathcal{T}}$ of coherent functors is a Frobenius abelian category such that the class of its projective-injective objects equals*

$$\text{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}} = \text{add}(\mathbf{p}(\mathcal{T})).$$

Proof. It is already observed that $\widehat{\mathcal{T}} \subseteq (\mathcal{T}^{\text{op}}, \text{Ab})$ is an abelian subcategory which has enough projective objects. The class of projective objects equals $\text{add}(\mathbf{p}(\mathcal{T}))$. Note that $(-, C)$ is a cohomological functor. By Lemma 2.3, we have, for each $F \in \widehat{\mathcal{T}}$ and $C \in \mathcal{T}$, $\text{Ext}^i(F, (-, C)) = 0$, $i \geq 1$. Therefore, the representable functor $(-, C)$ is an injective object in $\widehat{\mathcal{T}}$. Hence we have shown that $\widehat{\mathcal{T}}$ has enough projective objects and that every projective object is injective.

We claim that $\widehat{\mathcal{T}}$ has enough injective objects and that every injective object is projective. To see this, assume that F is a coherent functor, and take a presentation $(-, Y) \overset{(-, v)}{\to} (-, Z) \to F \to 0$. Then we have seen that there is a monomorphism $F \to (-, X[1])$ via the two-sided resolution of F. By the above proof, the functor $(-, X[1])$ is injective. Then the category $\widehat{\mathcal{T}}$ has enough injective objects. Assume further that F is an injective object in $\widehat{\mathcal{T}}$. Then the monomorphism $F \to (-, X[1])$ splits, and hence F is projective. This proves the claim, and then we have shown that $\widehat{\mathcal{T}}$ is a Frobenius category and the class of its projective-injective equals $\text{add}(\mathbf{p}(\mathcal{T}))$.

It remains to show $\text{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}} = \text{add}(\mathbf{p}(\mathcal{T}))$. We have already noticed that $\text{add}(\mathbf{p}(\mathcal{T})) \subseteq \text{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}}$. Let $H \in \text{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}}$. By Lemma 2.3, $\text{Ext}^i(F, H) = 0$ for all $F \in \widehat{\mathcal{T}}$ and $i \geq 0$. Then H is an injective object in $\widehat{\mathcal{T}}$, and hence by above H lies in $\text{add}(\mathbf{p}(\mathcal{T}))$. $\qquad\square$

In what follows R will be a commutative artinian ring. We denote by R-mod the category of finitely generated R-modules. The triangulated category $\mathcal{T}$ will be R-linear which is Hom-finite and Krull-Schmidt. Consider the category $(\mathcal{T}^{\text{op}}, R\text{-mod})$ of R-linear functors from $\mathcal{T}$ to R-mod. Observe that $(\mathcal{T}^{\text{op}}, R\text{-mod})$ is viewed as a full subcategory of $(\mathcal{T}^{\text{op}}, \text{Ab})$ via the forgetful functor. Recall the notion of generalized Serre duality from the introduction.

Proposition 2.5. *The full subcategories $\mathcal{T}_r$ and $\mathcal{T}_l$ are thick triangulated subcategories of $\mathcal{T}$.*

Proof. We only prove the result on $\mathcal{T}_r$. We first claim that

$$\mathcal{T}_r = \{X \in \mathcal{T} \mid D(X,-) \in \widehat{\mathcal{T}}\}.$$

To see this, note that $\mathcal{T}$ is Krull-Schmidt, hence it is idempotent-split; see [12] and [9, Theorem A.1]. Then we have $\mathrm{add}(\mathbf{p}(\mathcal{T})) = \mathbf{p}(\mathcal{T})$. Note that the functor $D(X,-)$ is always cohomological. Hence if $D(X,-)$ is coherent, then by Proposition 2.4, $D(X,-) \in \mathrm{Coho}(\mathcal{T}) \cap \widehat{\mathcal{T}} = \mathrm{add}(\mathbf{p}(\mathcal{T})) = \mathbf{p}(\mathcal{T})$, that is, $D(X,-)$ is representable. Hence, the functor $D(X,-)$ is coherent if and only if it is representable. Now the claim follows.

Note that $D(X[n],-) \simeq D(X,-) \circ [-n]$; here for each integer n, $[n]$ denotes the n-th power of the translation functor $[1]$ on $\mathcal{T}$. Then the subcategory $\mathcal{T}_r \subseteq \mathcal{T}$ is closed under the functors $[n]$. Let $X' \to X \to X'' \to X'[1]$ be a distinguished triangle such that $X', X'' \in \mathcal{T}_r$. Then we have an exact sequence of functors

$$(X'[1],-) \longrightarrow (X'',-) \longrightarrow (X,-) \longrightarrow (X',-) \longrightarrow (X''[-1],-),$$

and hence the following sequence

$$D(X''[-1],-) \longrightarrow D(X',-) \longrightarrow D(X,-) \longrightarrow D(X'',-) \longrightarrow D(X'[1],-)$$

is also exact. Since $X', X'' \in \mathcal{T}_r$, the four terms other than $D(X,-)$ in the above sequence lie in $\widehat{\mathcal{T}}$. By Lemmas 2.1 and 2.2 $\widehat{\mathcal{T}} \subseteq (\mathcal{T}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian subcategory which is closed under extensions. Hence we infer that $D(X,-) \in \widehat{\mathcal{T}}$, and then by the claim above, we have $X \in \mathcal{T}_r$. We have shown that $\mathcal{T}_r$ is a triangulated subcategory of $\mathcal{T}$. Observe that $\mathcal{T}_r$ is thick, that is, it is closed under taking direct summands, since direct summands of a coherent functor are still coherent. $\square$

Let us assume that $\mathcal{C}$ is a Hom-finite R-linear additive category which is *skeletally small*, that is, the iso-classes of objects in $\mathcal{C}$ form a set. Consider the full subcategory $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ of $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ consisting of R-linear functors. Then we have duality functors

$$D\colon (\mathcal{C}^{\mathrm{op}}, R\text{-mod}) \longrightarrow (\mathcal{C}, R\text{-mod}) \text{ and } D\colon (\mathcal{C}^{\mathrm{op}}, R\text{-mod}) \longrightarrow (\mathcal{C}, R\text{-mod}),$$

which are induced by the Matlis duality on R-mod. The additive category $\mathcal{C}$ is called a *dualizing R-variety* if the two duality functors D preserve coherent functors ([4]). It is easy to see that $\mathcal{C}$ is a dualizing R-variety if and only if $\mathcal{C}$ has pseudokernels and pseudocokernels, and $D(X,-) \in \widehat{\mathcal{C}}$ and $D(-,X) \in \widehat{\mathcal{C}^{\mathrm{op}}}$ for each object $X \in \mathcal{C}$; one may apply Lemma 2.1 and [4, Theorem 2.4(1)].

Recall from Appendix A.1 that a Hom-finite category $\mathcal{C}$ is said to have Serre duality, if $\mathcal{C}_r = \mathcal{C} = \mathcal{C}_l$. Combining Lemma 2.2 with the claim (and its dual) in the above proof, we get the following observation; compare [17, Proposition 2.11].

Corollary 2.6. *Let $\mathcal{T}$ be a skeletally small Hom-finite Krull-Schmidt triangulated category. Then $\mathcal{T}$ has Serre duality if and only if $\mathcal{T}$ is a dualizing R-variety.* $\square$

The following result generalizes slightly a result due to Bondal and Kapranov ([7, Proposition 3.3]). Let us stress that the tricky proof is modified from an argument independently due to Huybrechts ([16, Proposition 1.46]) and Van den Bergh ([6, Theorem A.4.4]). Recall the notions of generalized Serre duality and trace function in the appendix. For the notion of triangle functor, we refer to [18, section 8].

Proposition 2.7. *Let $\mathcal{T}$ a Hom-finite Krull-Schmidt triangulated category. Then there is a natural isomorphism $\eta_X \colon S(X[1]) \to S(X)[1]$ for each $X \in \mathcal{T}_r$ such that the pair (S, η) is a triangle functor between the triangulated categories $\mathcal{T}_r$ and $\mathcal{T}_l$.*

Proof. Recall from Proposition 2.5 that both $\mathcal{T}_r$ and $\mathcal{T}_l$ are triangulated categories of $\mathcal{T}$. We will construct a natural isomorphism

$$\eta_X \colon S(X[1]) \longrightarrow S(X)[1]$$

for each $X \in \mathcal{T}_r$, such that the pair (S, η) is a triangle functor.

Let $X \in \mathcal{T}_r$ and $Y \in \mathcal{T}$. Consider the following composite of natural R-linear isomorphisms

$$\Psi_{X,Y} \colon\; (Y, S(X[1])) \overset{\phi^{-1}_{X[1],Y}}{\longrightarrow} D(X[1], Y) \overset{-D\circ[-1]}{\longrightarrow} D(X, Y[-1])$$
$$\overset{\phi_{Y[-1],X}}{\longrightarrow} (Y[-1], S(X)) \overset{[1]}{\longrightarrow} (Y, S(X)[1]),$$

where ϕ is the natural isomorphism in Proposition A.1. Note the minus sign in the second isomorphism above. By following the composition carefully, we obtain that, for each $f \in (X, Y[-1])$ and $g \in (Y, S(X[1]))$,

$$(f,\; \Psi_{X,Y}(g)[-1])_{X,Y[-1]} = -(f[1],\; g)_{X[1],Y}.$$

Note that Ψ defines an isomorphism $\Psi_X \colon (-, S(X[1])) \simeq (-, S(X)[1])$ of R-linear functors. By Yoneda Lemma there is a unique isomorphism $\eta_X \colon S(X[1]) \to S(X)[1]$ such that $\Psi_X = (-, \eta_X)$, in other words, $\Psi_{X,Y}(g) = \eta_X \circ g$. Consequently, we have

(2.1) $$(f, (\eta_X \circ g)[-1])_{X,Y[-1]} = -(f[1], g)_{X[1],Y}.$$

We observe that $\Psi_{X,Y}$ is natural both in X and Y. It follows then that η_X is natural in X. In other words, the isomorphisms η_X define a natural isomorphism of functors.

Next we show that the pair (S, η) is a triangle functor. Given a distinguished triangle $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$ in $\mathcal{T}_r$. We have to show that the following triangle is also distinguished (in $\mathcal{T}_l$, or in $\mathcal{T}$)

$$S(X) \overset{S(u)}{\longrightarrow} S(Y) \overset{S(v)}{\longrightarrow} S(Z) \overset{\eta_X \circ S(w)}{\longrightarrow} S(X)[1].$$

We claim that the following sequence of functors is exact

$$(-, S(Y)) \longrightarrow (-, S(Z)) \longrightarrow (-, S(X)[1]) \longrightarrow (-, S(Y)[1]) \longrightarrow (-, S(Z)[1]).$$

In fact, from the distinguished triangle $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$, we infer that the following sequence is exact

$$(Z[1], -) \longrightarrow (Y[1], -) \longrightarrow (X[1], -) \longrightarrow (Z, -) \longrightarrow (Y, -),$$

and hence the following sequence is also exact

$$D(Y, -) \longrightarrow D(Z, -) \longrightarrow D(X[1], -) \longrightarrow D(Y[1], -) \longrightarrow D(Z[1], -).$$

Note that in the following commutative diagram of functors all vertical morphisms are isomorphisms

$$
\begin{array}{ccccccccc}
D(Y,-) & \longrightarrow & D(Z,-) & \longrightarrow & D(X[1],-) & \longrightarrow & D(Y[1],-) & \longrightarrow & D(Z[1],-) \\
\downarrow{\scriptstyle\phi_Y} & & \downarrow{\scriptstyle\phi_Z} & & \downarrow{\scriptstyle\Psi_X\circ\phi_{X[1]}} & & \downarrow{\scriptstyle\Psi_Y\circ\phi_{Y[1]}} & & \downarrow{\scriptstyle\Psi_Z\circ\phi_{Z[1]}} \\
(-, S(Y)) & \longrightarrow & (-, S(Z)) & \longrightarrow & (-, S(X)[1]) & \longrightarrow & (-, S(Y)[1]) & \longrightarrow & (-, S(Z)[1]).
\end{array}
$$

Here $\phi_{X[1]}$, $\phi_{Y[1]}$ and $\phi_{Z[1]}$ are the natural isomorphisms of functors induced by ϕ in Proposition A.1. This proves the claim.

Take a distinguished triangle $S(X) \overset{S(u)}{\to} S(Y) \overset{\alpha}{\to} W \overset{\beta}{\to} S(X)[1]$. We claim that to prove the result, it suffices to show that there is a morphism $\delta\colon W \to S(Z)$ such that the following diagram commutes.

$$
\begin{array}{ccccccc}
S(X) & \xrightarrow{S(u)} & S(Y) & \xrightarrow{\alpha} & W & \xrightarrow{\beta} & S(X)[1] \\
\Big\| & & \Big\| & & \Big\downarrow{\delta} & & \Big\| \\
S(X) & \xrightarrow{S(u)} & S(Y) & \xrightarrow{S(v)} & S(Z) & \xrightarrow{\eta_X \circ S(w)} & S(X)[1]
\end{array}
$$

In fact, assume that we already have the above commutative diagram. Then we have the following commutative diagram of functors

$$
\begin{array}{ccccccccc}
(-, S(X)) & \longrightarrow & (-, S(Y)) & \longrightarrow & (-, W) & \longrightarrow & (-, S(X)[1]) & \longrightarrow & (-, S(Y)[1]) \\
\Big\| & & \Big\| & & \Big\downarrow{(-,\delta)} & & \Big\| & & \Big\| \\
(-, S(X)) & \longrightarrow & (-, S(Y)) & \longrightarrow & (-, S(Z)) & \longrightarrow & (-, S(X)[1]) & \longrightarrow & (-, S(Y)[1])
\end{array}
$$

Note the the rows are exact (see the previous claim). Then Five Lemma implies that $(-, \delta)$ is an isomorphism, and hence by Yoneda Lemma δ is an isomorphism. We are done.

To complete the proof, it suffices to find a required morphism δ. To find such a morphism $\delta \in (W, S(Z))$ is to solve the two equations $\delta \circ \alpha = S(v)$ and $\eta_X \circ S(w) \circ \delta = \beta$. By (2.1) we have

$$(2.2) \qquad \operatorname{Tr}_X((\eta_X \circ f)[-1]) = -\operatorname{Tr}_{X[1]}(f), \quad f \in (X[1], S(X[1])).$$

For the definition of Tr and the bilinear form $(-, -)$, see Appendix. By the non-degeneratedness of the bilinear form $(-, -)$, we infer that $\delta \circ \alpha = S(v)$ is equivalent to the equations $\operatorname{Tr}_Z(S(v) \circ x) = \operatorname{Tr}_Z(\delta \circ \alpha \circ x)$ for all $x \in (Z, S(Y))$; and that $\eta_X \circ S(w) \circ \delta = \beta$ is equivalent to the equations $\operatorname{Tr}_X((\eta_X \circ S(w) \circ \delta)[-1] \circ y) = \operatorname{Tr}_X(\beta[-1] \circ y)$ for all $y \in (X, W[-1])$.

Note that we have

$$
\begin{aligned}
\operatorname{Tr}_X((\eta_X \circ S(w) \circ \delta)[-1] \circ y) &= -\operatorname{Tr}_{X[1]}(S(w) \circ \delta \circ y[1]) \\
&= -\operatorname{Tr}_Z(\delta \circ y[1] \circ w),
\end{aligned}
$$

where the first equality uses (2.2) and the second uses (A.4). Hence it suffices to find $\delta \in (W, S(Z))$ such that

$$
\begin{aligned}
\operatorname{Tr}_Z(\delta \circ \alpha \circ x) &= \operatorname{Tr}_Z(S(v) \circ x), \quad \forall\, x \in (Z, S(Y)), \\
\operatorname{Tr}_Z(\delta \circ y[1] \circ w) &= -\operatorname{Tr}_X(\beta[-1] \circ y), \quad \forall\, y \in (X, W[-1]).
\end{aligned}
$$

By Proposition A.1 and its proof we have the isomorphism $\phi_{W,Z}\colon (W, S(Z)) \simeq D(Z, W)$ and the equality $\phi_{W,Z}(\delta) = \operatorname{Tr}_Z(\delta \circ -)$. Here we recall that E is the minimal injective cogenerator for R and that $D = \operatorname{Hom}_R(-, E)$. Hence to complete the proof, it suffices to find an R-linear morphism F from (Z, W) to E such that

$$
\begin{aligned}
F(\alpha \circ x) &= \operatorname{Tr}_Z(S(v) \circ x), \quad \forall\, x \in (Z, S(Y)), \\
F(y[1] \circ w) &= -\operatorname{Tr}_X(\beta[-1] \circ y), \quad \forall\, y \in (X, W[-1]).
\end{aligned}
$$

Using the injectivity of E, it is not hard to see that such a morphism F exists provided that whenever $\alpha \circ x = y[1] \circ w$, then $\mathrm{Tr}_Z(S(v) \circ x) = -\mathrm{Tr}_X(\beta[-1] \circ y)$. Now we assume that $\alpha \circ x = y[1] \circ w$. Then we have the following morphism of distinguished triangles.

$$
\begin{array}{ccccccc}
Y & \xrightarrow{\ v\ } & Z & \xrightarrow{\ w\ } & X[1] & \xrightarrow{-u[1]} & Y[1] \\
\Big\downarrow{\scriptstyle\phi_0} & & \Big\downarrow{\scriptstyle x} & & \Big\downarrow{\scriptstyle y[1]} & & \Big\downarrow{\scriptstyle\phi_0[1]} \\
S(X) & \xrightarrow{S(u)} & S(Y) & \xrightarrow{\ \alpha\ } & W & \xrightarrow{\ \beta\ } & S(X)[1]
\end{array}
$$

Therefore we have

$$
\begin{aligned}
\mathrm{Tr}_Z(S(v) \circ x) &= \mathrm{Tr}_Y(x \circ v) \\
&= \mathrm{Tr}_Y(S(u) \circ \phi_0) \\
&= \mathrm{Tr}_X(\phi_0 \circ u) \\
&= \mathrm{Tr}_X(-\beta[-1] \circ y) \\
&= -\mathrm{Tr}_X(\beta[-1] \circ y),
\end{aligned}
$$

where the first and third equality use (A.4) and the second and fourth use the commutativity of the diagram above. This completes the proof. □

Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category as above. Recall from [12] and [13, 1.2] that an *Auslander-Reiten triangle* in $\mathcal{T}$ is a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ such that both X and Z are indecomposable, and that $w \neq 0$ satisfies that for each non-retraction $\gamma \colon Y' \to Z$ in $\mathcal{T}$ we have $w \circ \gamma = 0$. By [12, p.31] the morphism u is left almost-split, that is, any non-section $h \colon X \to Y'$ factors through u; dually, the morphism v is right almost-split.

The following result is a slight generalization of a result due to Reiten and Van den Bergh ([26, Proposition I.2.3]), where the result is formulated in a different terminology.

Proposition 2.8. *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category, and let $X \in \mathcal{T}$ be an indecomposable object. Then we have*

(1) *$X \in \mathcal{T}_r$ if and only if there is an Auslander-Reiten triangle with X as the third term;*

(2) *$X \in \mathcal{T}_l$ if and only if there is an Auslander-Reiten triangle with X as the first term.*

Proof. We only prove the first statement. By a dual argument, one proves the second one.

For the "only if" part, assume that $X \in \mathcal{T}_r$ is indecomposable. Recall from Proposition A.1 that the generalized Serre functor $S \colon \mathcal{T}_r \to \mathcal{T}_l$ is an equivalence. Therefore $S(X)$ is also indecomposable. For each $X \in \mathcal{T}_r$ and $Y \in \mathcal{T}$ there is a non-degenerated bilinear form $(-,-)_{X,Y} \colon (X,Y) \times (Y,S(X)) \to E$; see the proof of Proposition A.1. Take a nonzero morphism $w \colon X \to S(X)$ such that $(\mathrm{rad}\,\mathrm{End}_{\mathcal{T}}(X), w)_{X,X} = 0$. Here $\mathrm{rad}\,\mathrm{End}_{\mathcal{T}}(X)$ denotes the Jacobson radical of $\mathrm{End}_{\mathcal{T}}(X)$. Then we form a distinguished triangle $S(X)[-1] \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{w} S(X)$. We claim that it is an Auslander-Reiten triangle. Then we are done.

In fact, it suffices to show that each morphism $\gamma \colon X' \to X$, which is not split-epic, satisfies $w \circ \gamma = 0$. To see this, for any $x \in (X, X')$, consider $(x, w \circ \gamma)_{X,X'} =$

$(\gamma \circ x, w)_{X,X}$; see (A.2). Since γ is a not split-epic, then $\gamma \circ x \in \text{rad End}_{\mathcal{T}}(X)$, and thus $(\gamma \circ x, w)_{X,X} = 0$. Therefore $(x, w \circ \gamma)_{X,X'} = 0$. By the non-degeneratedness of the bilinear form, we have $w \circ \gamma = 0$.

For the "if" part, assume that we have an Auslander-Reiten triangle $Z \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{w} Z[1]$. Take any R-linear morphism $\text{Tr}: (X, Z[1]) \to E$ such that $\text{Tr}(w) \neq 0$. Here we use that E is an injective cogenerator. We claim that for each $X' \in \mathcal{T}$ the pairing

$$(-, -) : (X, X') \times (X', Z[1]) \longrightarrow E$$

given by $(f, \ g) = \text{Tr}(g \circ f)$ is non-degenerated. If so, we have an induced isomorphism of R-linear functors $D(X, -) \simeq (-, Z[1])$. Hence $X \in \mathcal{T}_r$, and then we are done.

To prove the claim, let $X' \in \mathcal{T}$. Let $f : X \to X'$ be a nonzero morphism, and let $X'' \xrightarrow{s} X \xrightarrow{f} X' \to X''[1]$ be a distinguished triangle. Since $f \neq 0$, then s is not split-epic. Then by the properties of Auslander-Reiten triangle, we have $w \circ s = 0$. This implies that w factors through f, that is, there is a morphism $g : X' \to Z[1]$ such that $w = g \circ f$, and hence $(f, \ g) \neq 0$. On the other hand, let $g : X' \to Z[1]$ be a nonzero morphism, and let $Z \xrightarrow{u'} X''' \to X' \xrightarrow{g} Z[1]$ be a distinguished triangle. Since $g \neq 0$, then u' is not split-mono. Then by the properties of Auslander-Reiten triangle, we deduce that u' factors thorough u, and then we have the following morphism of distinguished triangles

$$
\begin{array}{ccccccc}
Z & \xrightarrow{u} & Y & \xrightarrow{v} & X & \xrightarrow{w} & Z[1] \\
\Big\| & & \Big\downarrow & & \Big\downarrow{f} & & \Big\| \\
Z & \xrightarrow{u'} & X''' & \longrightarrow & X' & \xrightarrow{g} & Z[1]
\end{array}
$$

Hence $g \circ f = w$ and then $(f, \ g) \neq 0$. This completes the proof. $\square$

3. Computing generalized Serre duality

In this section, we will compute explicitly the generalized Serre duality for the bounded derived categories of artin algebras and of certain noncommutative projective schemes in the sense of Artin and Zhang ([1]); see Theorem 3.5 and Theorem 3.10.

Let R be a commutative artinian ring. Let $\mathcal{C}$ be an R-linear category which is Hom-finite and which is assumed to be additive. Denote by $S_{\mathcal{C}} : \mathcal{C}_r \to \mathcal{C}_l$ its generalized Serre functor. Denote by $(-, -)_{X,Y} : (X, Y) \times (Y, S_{\mathcal{C}}(X)) \to E$ the associated bilinear form for each $X \in \mathcal{C}_r$ and $Y \in \mathcal{C}$. Here E is the minimal injective cogenerator of R. For details, see Appendix.

Let $\mathbf{K}^b(\mathcal{C})$ be the bounded homotopy category of complexes in $\mathcal{C}$. Note that the category $\mathbf{K}^b(\mathcal{C})$ is naturally R-linear and then is Hom-finite. Denote its generalized Serre functor by $S : \mathbf{K}^b(\mathcal{C})_r \to \mathbf{K}^b(\mathcal{C})_l$. The following result is inspired by the argument in [12, p.37].

Proposition 3.1. *Use the notation as above. Then we have*

$$\mathbf{K}^b(\mathcal{C}_r) \subseteq \mathbf{K}^b(\mathcal{C})_r, \ \mathbf{K}^b(\mathcal{C}_l) \subseteq \mathbf{K}^b(\mathcal{C})_l \ \text{and} \ S|_{\mathbf{K}^b(\mathcal{C}_r)} = S_{\mathcal{C}},$$

where $S_{\mathcal{C}}$ is viewed as a functor on complexes between $\mathbf{K}^b(\mathcal{C}_r)$ and $\mathbf{K}^b(\mathcal{C}_l)$.

Proof. To prove the result, it suffices to build up a natural non-degenerated bilinear form

$$(-,-)_{X^\bullet, Y^\bullet} \colon \mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(X^\bullet, Y^\bullet) \times \mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(Y^\bullet, S_\mathcal{C}(X^\bullet)) \longrightarrow E$$

for each $X^\bullet \in \mathbf{K}^b(\mathcal{C}_r)$ and $Y^\bullet \in \mathbf{K}^b(\mathcal{C})$. We define

$$(f^\bullet, g^\bullet)_{X^\bullet, Y^\bullet} = \sum_{i \in \mathbb{Z}} (-1)^i (f^i, g^i)_{X^i, Y^i},$$

where $f^\bullet = (f^i)$ and $g^\bullet = (g^i)$ are chain morphisms.

Note that the bilinear form is well defined. In fact, if $f^\bullet$ is homotopic to 0, then $(f^\bullet, g^\bullet)_{X^\bullet, Y^\bullet} = 0$. Assume that $f^i = d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i$ for each $i \in \mathbb{Z}$. Here $\{h^i \colon X^i \to Y^{i-1}\}_{i \in \mathbb{Z}}$ is the homotopy. Then we have

$$
\begin{aligned}
(f^\bullet,\, g^\bullet)_{X^\bullet, Y^\bullet} &= \sum_{i \in \mathbb{Z}} (-1)^i (d_Y^{i-1} \circ h^i,\, g^i)_{X^i, Y^i} + (-1)^i (h^{i+1} \circ d_X^i,\, g^i)_{X^i, Y^i} \\
&= \sum_{i \in \mathbb{Z}} (-1)^i (h^i,\, g^i \circ d_Y^{i-1})_{X^i, Y^{i-1}} + (-1)^i (h^{i+1},\, S_\mathcal{C}(d_X^i) \circ g^i)_{X^{i+1}, Y^i} \\
&= \sum_{i \in \mathbb{Z}} (-1)^i (h^i,\, g^i \circ d_Y^{i-1} - d_{S_\mathcal{C}(X)}^{i-1} \circ g^{i-1})_{X^i, Y^{i-1}} \\
&= 0.
\end{aligned}
$$

The second equality follows from the identities (A.1) and (A.2) in the appendix, and the last one follows from the fact that $g^\bullet \colon Y^\bullet \to S_\mathcal{C}(X^\bullet)$ is a chain morphism.

To prove the naturalness of the bilinear form is to verify the following two identities

$$
\begin{aligned}
(f^\bullet \circ \theta^\bullet,\, g^\bullet)_{X'^\bullet, Y^\bullet} &= (f^\bullet,\, S_\mathcal{C}(\theta^\bullet) \circ g^\bullet)_{X^\bullet, Y^\bullet} \\
(\gamma^\bullet \circ f^\bullet,\, g^\bullet)_{X^\bullet, Y^\bullet} &= (f^\bullet,\, g^\bullet \circ \gamma^\bullet)_{X^\bullet, Y'^\bullet}.
\end{aligned}
$$

Note that they can be easily derived from the identities (A.1) and (A.2) for the category $\mathcal{C}$.

We will prove the non-degeneratedness. For this end, fix a complex $X^\bullet \in \mathbf{K}^b(\mathcal{C}_r)$. Denote by

$$\phi \colon D\mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(X^\bullet, -) \longrightarrow \mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(-, S_\mathcal{C}(X^\bullet))$$

the natural transformation induced from the above bilinear form. We claim that ϕ is a natural isomorphism. Then we are done.

In fact, since ϕ is a natural transformation between two cohomological functors, so by devissage, it suffices to prove that

$$\phi_{Y^\bullet} \colon D\mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(X^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(Y^\bullet, S_\mathcal{C}(X^\bullet))$$

is an isomorphism for each stalk complex $Y^\bullet$. Without loss of generality, assume that $Y^\bullet = Y$ for some object $Y \in \mathcal{C}$. Then $D\mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(X^\bullet, Y)$ is just the zeroth cohomology of the following complex

$$\cdots \longrightarrow D(X^{-1}, Y) \longrightarrow D(X^0, Y) \longrightarrow D(X^1, Y) \longrightarrow \cdots$$

By the generalized Serre duality of $\mathcal{C}$, the above complex isomorphic to the following complex

$$\cdots \longrightarrow (Y, S_\mathcal{C}(X^{-1})) \longrightarrow (Y, S_\mathcal{C}(X^0)) \longrightarrow (Y, S_\mathcal{C}(X^1)) \longrightarrow \cdots$$

While the zeroth cohomology of this complex is nothing but $\mathrm{Hom}_{\mathbf{K}^b(\mathcal{C})}(Y, S_\mathcal{C}(X^\bullet))$. This proves that ϕ_Y is an isomorphism. $\square$

Let $\mathcal{A}$ be an R-linear abelian category. Denote by $\mathbf{D}^b(\mathcal{A})$ its bounded derived category, which is naturally R-linear. We will assume that $\mathbf{D}^b(\mathcal{A})$ is Hom-finite. In this case the abelian category $\mathcal{A}$ is said to be *Ext-finite* ([20]). Observe that an Ext-finite abelian category is Hom-finite. By [8, Corollary 2.10] the category $\mathbf{D}^b(\mathcal{A})$ is idempotent-split, hence by [9, Theorem A.2] it is Krull-Schmidt; also see [20, Corollary B]. Denote by $\mathcal{P}$ and $\mathcal{I}$ the full subcategory of $\mathcal{A}$ consisting of projective objects and injective objects, respectively. Let $S_{\mathcal{A}} \colon \mathcal{A}_r \to \mathcal{A}_l$ be the generalized Serre functor on $\mathcal{A}$.

We claim that $\mathcal{A}_r \subseteq \mathcal{P}$ and $\mathcal{A}_l \subseteq \mathcal{I}$. Take an object $X \in \mathcal{A}_r$. By the generalized Serre duality, we have an isomorphism of functors $(X, -) \simeq D(-, S_{\mathcal{A}}(X))$. Hence the functor $(-, X)$ is right exact, and then the object X is projective. Similarly each object in $\mathcal{A}_l$ is injective.

Note that we have natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(P^\bullet, X^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}^b(\mathcal{A})}(P^\bullet, X^\bullet), \quad \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, I^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}^b(\mathcal{A})}(X^\bullet, I^\bullet),$$

for any $P^\bullet \in \mathbf{K}^b(\mathcal{P})$, $X^\bullet \in \mathbf{K}^b(\mathcal{A})$ and $I^\bullet \in \mathbf{K}^b(\mathcal{I})$. In particular, $\mathbf{K}^b(\mathcal{P})$ and $\mathbf{K}^b(\mathcal{I})$ are naturally viewed as triangulated subcategories of $\mathbf{D}^b(\mathcal{A})$; compare [14, 1.4] and [18, Example 12.2].

We have the following direct consequence of Proposition 3.1.

Corollary 3.2. *Let $\mathcal{A}$ be an Ext-finite abelian category. Then we have*

$$\mathbf{K}^b(\mathcal{A}_r) \subseteq \mathbf{D}^b(\mathcal{A})_r, \ \mathbf{K}^b(\mathcal{A}_l) \subseteq \mathbf{D}^b(\mathcal{A})_l \text{ and } S|_{\mathbf{K}^b(\mathcal{A}_r)} = S_{\mathcal{A}},$$

where S denotes the generalized Serre functor on $\mathbf{D}^b(\mathcal{A})$. $\qquad\qquad\square$

Following [24, Appendix] a complex $X^\bullet$ in $\mathbf{D}^b(\mathcal{A})$ is said to *have finite projective dimension*, if there exists some integer N such that $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, M[n]) = 0$ for any $M \in \mathcal{A}$ and $n \geq N$. Set $\mathbf{D}^b(\mathcal{A})_{\mathrm{fpd}}$ to be the full subcategory of $\mathbf{D}^b(\mathcal{A})$ consisting of complexes of finite projective dimension; it is a thick triangulated subcategory.

The following is well known; compare the proof of [27, Proposition 6.2].

Lemma 3.3. *We have $\mathbf{K}^b(\mathcal{P}) \subseteq \mathbf{D}^b(\mathcal{A})_{\mathrm{fpd}}$. On the other hand, if the category $\mathcal{A}$ has enough projective objects, then $\mathbf{K}^b(\mathcal{P}) = \mathbf{D}^b(\mathcal{A})_{\mathrm{fpd}}$.* $\qquad\square$

Dually one defines a complex of *finite injective dimension* and introduces the triangulated subcategory $\mathbf{D}^b(\mathcal{A})_{\mathrm{fid}}$ of $\mathbf{D}^b(\mathcal{A})$. Then one has the dual version of Lemma 3.3; compare [15, Chapter I, Proposition 7.6].

We have the following observation.

Proposition 3.4. *Let $\mathcal{A}$ be an Ext-finite abelian category. Then we have $\mathbf{D}^b(\mathcal{A})_r \subseteq \mathbf{D}^b(\mathcal{A})_{\mathrm{fpd}}$ and $\mathbf{D}^b(\mathcal{A})_l \subseteq \mathbf{D}^b(\mathcal{A})_{\mathrm{fid}}$.*

Proof. Take a complex $X^\bullet \in \mathbf{D}^b(\mathcal{A})_r$. Then we have an isomorphism of functors over $\mathbf{D}^b(\mathcal{A})$: $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, -) \simeq D\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(-, S(X^\bullet))$, where S is the generalized Serre functor on $\mathbf{D}^b(\mathcal{A})$. Take a positive integer N such that $H^n(S(X^\bullet)) = 0$ for any $n \leq -N$; here $H^n(-)$ denotes the n-th cohomology of a complex. Let $M \in \mathcal{A}$ and let $n \geq N$. Then $\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, M[n]) \simeq D\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(M[n], S(X^\bullet)) = 0$. Hence $X^\bullet$ has finite projective dimension. Then we get the first inclusion, and the second one can be proved dually. $\qquad\square$

Here is our first explicit result on generalized Serre duality.

Theorem 3.5. *Let $\mathcal{A}$ be a Hom-finite abelian category with enough projective and enough injective objects. Assume that $\mathcal{A}_r = \mathcal{P}$ and $\mathcal{A}_l = \mathcal{I}$. Denote the generalized Serre functor $S_{\mathcal{A}}$ by ν which is called the Nakayama functor. Then we have $\mathbf{D}^b(\mathcal{A})_r = \mathbf{K}^b(\mathcal{P})$, $\mathbf{D}^b(\mathcal{A})_l = \mathbf{K}^b(\mathcal{I})$ and the generalized Serre functor on $\mathbf{D}^b(\mathcal{A})$ is given by ν which operates on complexes term by term.*

Proof. Note that under the assumption, the abelian category $\mathcal{A}$ is Ext-finite. Then the result follows immediately from Corollary 3.2, Lemma 3.3 and Proposition 3.4. □

We have the following motivating example of Theorem 3.5.

Example 3.6. (Happel) *Let A be an artin R-algebra. Denote by A-mod the category of finitely generated left A-modules. We denote by A-proj (resp. A-inj) the full subcategory consisting of projective (resp. injective) modules.*

It is well known that the category A-mod satisfies the assumption of Theorem 3.5. In this case, the functor ν is the usual Nakayama functor $D\mathrm{Hom}_A(-, A)$; see [12, p.37]. We apply Theorem 3.5 to conclude that $\mathbf{D}^b(A\text{-mod})_r = \mathbf{K}^b(A\text{-proj})$, $\mathbf{D}^b(A\text{-mod})_l = \mathbf{K}^b(A\text{-inj})$ and the generalized Serre functor is given by ν. We apply this explicit computational result. We infer from the statement (2) in Main Theorem that in an Auslander-Reiten triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ in $\mathbf{D}^b(A\text{-mod})$ we have $X^\bullet \in \mathbf{K}^b(A\text{-inj})$ and $Z^\bullet \in \mathbf{K}^b(A\text{-proj})$. Moreover, we have $X^\bullet \simeq \nu(Z^\bullet)$; see the proof of Proposition 2.8. This observation is essentially due to Happel ([13, Theorem 1.4]).

In what follows we will consider the conditions (C) and (C') which are introduced in Appendix A.2. We have the following observation.

Lemma 3.7. *Let $\mathcal{A}$ be a Hom-finite abelian category. Then we have*

(1) *if $\mathcal{A}$ has enough projective objects and $\mathcal{A}_r = \mathcal{P}$, then $\mathbf{D}^b(\mathcal{A})$ satisfies the condition (C);*

(2) *if $\mathcal{A}$ has enough injective objects and $\mathcal{A}_l = \mathcal{I}$, then $\mathbf{D}^b(\mathcal{A})$ satisfies the condition (C').*

Proof. We only prove (1). Assume that $\mathcal{A}$ has enough projective objects and that $\mathcal{A}_r = \mathcal{P}$. By Corollary 3.2, Lemma 3.3 and Proposition 3.4, we have $\mathbf{D}^b(\mathcal{A})_r = \mathbf{K}^b(\mathcal{P})$.

Consider complexes $X^\bullet, X^{\bullet\prime} \in \mathbf{D}^b(\mathcal{A})_r$ and $Z^\bullet \in \mathbf{D}^b(\mathcal{A})$. Since $\mathcal{A}$ has enough projective objects, we may take a quasi-isomorphism $P^\bullet \to Z^\bullet$ such that $P^\bullet$ is a bounded-above complex of projective objects with finitely many cohomologies; see [15, Chapter I, Lemma 4.6]. Let $n \gg 0$. Take $Z^{\bullet\prime} = \sigma^{\geq -n} P^\bullet$ to be the brutal truncation, and take s to be the following composite

$$Z^{\bullet\prime} \longrightarrow P^\bullet \longrightarrow Z^\bullet,$$

where $Z^{\bullet\prime} \to P^\bullet$ is the natural chain map. Observe that $Z^{\bullet\prime} \in \mathbf{D}^b(\mathcal{A})_r$. It is direct to check that the following two maps induced by s

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, Z^{\bullet\prime}) \to \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X^\bullet, Z^\bullet), \ \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(Z^{\bullet\prime}, X^{\bullet\prime}) \to \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(Z^\bullet, X^{\bullet\prime})$$

are isomorphisms. We are done with the condition (C). □

We will call an abelian category $\mathcal{A}$ a *Gorenstein category* provided that $\mathbf{D}^b(\mathcal{A})_{\mathrm{fpd}}$ coincides with $\mathbf{D}^b(\mathcal{A})_{\mathrm{fid}}$; compare [14]. By combining Theorem 3.5, Lemma 3.7 and Proposition A.4 together, we have the following result.

Proposition 3.8. *Let $\mathcal{A}$ be a Hom-finite abelian category with enough projective and enough injective objects. Assume that $\mathcal{A}_r = \mathcal{P}$ and $\mathcal{A}_l = \mathcal{I}$. Then the following statements are equivalent:*

   (1) *the category $\mathbf{K}^b(\mathcal{P})$ has Serre duality;*
   (2) *the abelian category $\mathcal{A}$ is Gorenstein;*
   (3) *the category $\mathbf{K}^b(\mathcal{I})$ has Serre duality.* $\qquad\qquad\square$

The following immediate consequence gives a characterization of Gorenstein algebras, which seems to be of independent interest. Recall that an artin R-algebra A is *Gorenstein* provided that the regular module A has finite injective dimension on both sides ([14]). We observe that an artin algebra A is Gorenstein if and only if the abelian category A-mod of finitely generated left A-modules is Gorenstein in the sense above.

Corollary 3.9. (compare [14, Theorem 3.4]) *Let A be an artin R-algebra. Then the following statements are equivalent:*

   (1) *the category $\mathbf{K}^b(A\text{-proj})$ has Serre duality;*
   (2) *the artin algebra A is Gorenstein;*
   (3) *the category $\mathbf{K}^b(A\text{-inj})$ has Serre duality.* $\qquad\qquad\square$

We will next consider the generalized Serre duality on the bounded derived category of coherent sheaves on certain noncommutative projective schemes in the sense of Artin and Zhang ([1]).

We will follow [24, section 2 and Appendix] closely. Recall some standard notation. Let k be a field. Let $A = \oplus_{n \geq 0} A_n$ be a positively graded k-algebra which is *connected*, that is, $A_0 \simeq k$. We will assume that A is left noetherian. Denote by A-Gr (*resp.* A-gr) the category of graded left A-modules (*resp.* finitely-generated graded left A-modules) with morphisms preserving degrees. For each $d \in \mathbb{Z}$, denote by (d) be the degree-shift functor on A-Gr and A-gr. For $j \geq 0$, denote by $\mathrm{Ext}^j_{A\text{-Gr}}(-, -)$ and $\underline{\mathrm{Ext}}^j(-, -)$ the Ext functors on the category A-Gr and the associated graded Ext functors, respectively. Then we have $\underline{\mathrm{Ext}}^j(M, N) = \oplus_{d \in \mathbb{Z}} \mathrm{Ext}^j_{A\text{-Gr}}(M, N(d))$ for graded modules M, N. We will view k as a graded A-module concentrated at degree zero.

For a graded A-module M denote by $\tau(M)$ the sum of all its finite dimensional graded submodules. This gives rise naturally to the torsion functor $\tau \colon A\text{-Gr} \to A\text{-Gr}$. A graded module M is called *a torsion module* provided that $\tau(M) = M$. Denote by A-Tor (*resp.* A-tor) the full subcategories of A-Gr (*resp.* A-gr) consisting of torsion modules. Note that both of them are Serre subcategories. One defines the quotient abelian categories

$$\mathrm{Tails}(A) = A\text{-Gr}/A\text{-Tor} \quad \text{and} \quad \mathrm{tails}(A) = A\text{-gr}/A\text{-tor},$$

which are called the category of *quasi-coherent* and *coherent sheaves* on the noncommutative projective scheme $\mathrm{Proj}(A)$, respectively. Observe that both categories are naturally k-linear.

A connected left noetherian algebra A is said to *satisfy the condition "χ"* provided that for each module $M \in A$-gr, the Ext spaces $\underline{\mathrm{Ext}}^j(k, M)$ are finite dimensional for all $j \geq 1$; compare [1, Definition 3.2]. In this case, the abelian category $\mathrm{tails}(A)$ is Ext-finite; see [24, Proposition 2.2.1].

Here is our second explicit result on generalized Serre duality. Let us remark that it is somehow a restatement of a result due to de Naeghel and Van den Bergh ([24, Theorem A.4]); compare [28, Proposition 7.48] for the commutative case. For the notion of balanced dualizing complex and other unexplained notions, we refer to [30] and [24].

Theorem 3.10. *Let A be a connected (two-sided) noetherian algebra over a field k. Assume that*

(1) *the algebra A satisfies "χ" and the functor τ is of finite cohomological dimension;*

(2) *the opposite algebra A^{op} satisfies "χ" and the functor τ^{op} is of finite cohomological dimension.*

Then we have $\mathbf{D}^b(\mathrm{tails}(A))_r = \mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fpd}}$, $\mathbf{D}^b(\mathrm{tails}(A))_l = \mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fid}}$ and the generalized Serre functor is given $S = - \otimes^{\mathbf{L}} \mathcal{R}^{\bullet}[-1]$, where $\mathcal{R}^{\bullet}$ is induced by the balanced dualizing complex $R^{\bullet}$ of A.

Proof. Note that the assumption implies the existence of a balanced dualizing complex $R^{\bullet}$ in the sense of [30]. Here the functor $- \otimes^{\mathbf{L}} \mathcal{R}^{\bullet}$ is induced by the functor $- \otimes^{\mathbf{L}} R^{\bullet}$. For details, see [24, Appendix].

Note that the abelian category $\mathrm{tails}(A)$ is Ext-finite. By Proposition 3.4, we have

$$\mathbf{D}^b(\mathrm{tails}(A))_r \subseteq \mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fpd}} \text{ and } \mathbf{D}^b(\mathrm{tails}(A))_l \subseteq \mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fid}}.$$

We apply [24, Theorem A.4]. We obtain $\mathbf{D}^b(\mathrm{tails}(A))_r \supseteq \mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fpd}}$; moreover, the restriction of the generalized Serre functor $S|_{\mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fpd}}}$ is isomorphic to $- \otimes^{\mathbf{L}} \mathcal{R}^{\bullet}[-1]$, which is an equivalence between $\mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fpd}}$ and $\mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fid}}$; here we implicitly use [1, Proposition 2.3] and [24, Lemma 2.2]. This equivalence implies that $\mathbf{D}^b(\mathrm{tails}(A))_{\mathrm{fid}} \subseteq \mathbf{D}^b(\mathrm{tails}(A))_l$, completing the proof. $\square$

We end this section with two remarks.

Remark 3.11. (1) Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category. We remark that the Verdier quotient triangulated categories $\mathcal{T}/\mathcal{T}_r$ and $\mathcal{T}/\mathcal{T}_l$ measure how far the triangulated category $\mathcal{T}$ is from having Serre duality. Observe that in the case of Example 3.6 the quotient category $\mathcal{T}/\mathcal{T}_r$ coincides with the *singularity category* of the artin algebra A ([25, 10]); similarly, in the (commutative) case of Theorem 3.10 the quotient category $\mathcal{T}/\mathcal{T}_r$ coincides with the singularity category of the corresponding (commutative) projective scheme.

(2) Note that there is an analogue of Reiten-Van den Bergh's theorem for abelian categories in [21]. Roughly speaking, a Hom-finite abelian category has the so-called *Auslander-Reiten duality* if and only if it has enough Auslander-Reiten sequences. We expect that there exists a reasonable notion of *generalized Auslander-Reiten duality* for an arbitrary Hom-finite abelian category. This generalized duality might be useful in the study of an abelian category which does not have enough Auslander-Reiten sequences. For example, the category of finite dimensional comodules over a coalgebra is often such an abelian category.

Appendix A. Generalized Serre duality on linear category

Throughout this appendix, let R be a commutative artinian ring. Let $\mathcal{C}$ be an R-linear category which is Hom-finite. We do not assume that $\mathcal{C}$ is an additive category. Use the notation as in the introduction. For example, for two objects X and Y, (X, Y) stands for $\operatorname{Hom}_{\mathcal{C}}(X, Y)$; $D = \operatorname{Hom}_R(-, E)$ is the Matlis duality, where E is the minimal injective cogenerator of R.

A.1.   In this subsection, we introduce the notion of generalized Serre duality on a linear category. One might compare with the treatment in [7, Proposition 3.4], [26, p.301] and [22, section 3].

Let $\mathcal{C}$ be a linear category as above. Recall the two full subcategories $\mathcal{C}_r$ and $\mathcal{C}_l$ in the introduction. We have the following basic result.

Proposition A.1. *Use the notation as above. Then there is a unique functor (up to isomorphism) $S \colon \mathcal{C}_r \to \mathcal{C}_l$ such that for each $X \in \mathcal{C}_r$ and $Y \in \mathcal{C}$, there is a natural R-linear isomorphism*

$$\phi_{X,Y} \colon D(X, Y) \simeq (Y, S(X)).$$

Moreover, the functor S is an R-linear equivalence between $\mathcal{C}_r$ and $\mathcal{C}_l$.

Proof. We will first construct the functor S. Take an object $X \in \mathcal{C}_r$. Then there is an object $S(X) \in \mathcal{C}$ such that there is an isomorphism of R-linear functors $\phi_X \colon D(X, -) \simeq (-, S(X))$. We stress that this isomorphism is required to preserve the R-linear structure.

Given any morphism $f \colon X \to Y$ in $\mathcal{C}_r$, define the morphism $S(f) \colon S(X) \to S(Y)$ such that the following diagram commutes

$$
\begin{array}{ccc}
D(X, -) & \xrightarrow{\;\phi_X\;} & (-, S(X)) \\
{\scriptstyle D(f,-)}\Big\downarrow & & \Big\downarrow{\scriptstyle (-,S(f))} \\
D(Y, -) & \xrightarrow{\;\phi_Y\;} & (-, S(Y))
\end{array}
$$

Here we apply Yoneda Lemma. Moreover, the morphism $S(f)$ is uniquely determined by the above commutative diagram. It is direct to check that $S \colon \mathcal{C}_r \to \mathcal{C}$ is a R-linear functor. Note that for each object $X \in \mathcal{C}_r$, we have an isomorphism of R-linear functors $D(-, S(X)) \simeq (X, -)$, and then we have $S(X) \in \mathcal{C}_l$. In other words, we have obtained an R-linear functor $S \colon \mathcal{C}_r \to \mathcal{C}_l$. By the construction of S, we observe that there is a natural R-linear isomorphism

$$\phi_{X,Y} \colon D(X, Y) \xrightarrow{\;\simeq\;} (Y, S(X))$$

for $X \in \mathcal{C}_r$ and $Y \in \mathcal{C}$. Here we put $\phi_{X,Y} = \phi_X(Y)$. The naturalness of $\phi_{X,Y}$ in Y is obvious, while its naturalness in X is a direct consequence of the action of the functor S on morphisms.

For the uniqueness of the functor S, assume that there is another functor $S' \colon \mathcal{C}_r \to \mathcal{C}_l$ such that there is a natural R-linear isomorphism $\phi'_{X,Y} \colon D(X, Y) \simeq (Y, S'(X))$. In particular, there is an isomorphism of R-linear functors $D(X, -) \simeq (-, S'(X))$. However by the above proof, we have seen $\phi_X \colon D(X, -) \simeq (-, S(X))$. Therefore $(-, S(X)) \simeq (-, S'(X))$, and hence Yoneda Lemma implies that there is a unique isomorphism $S(X) \simeq S'(X)$. It is routine to check that this gives rise to a natural isomorphism between the R-linear functors S and S'.

We will show that the functor $S\colon \mathcal{C}_r \to \mathcal{C}_l$ is fully faithful and dense, and then it is a R-linear equivalence of categories. For this end, we need to introduce some notation. For any $X \in \mathcal{C}_r$ and $Y \in \mathcal{C}$, define a non-degenerated bilinear form

$$(-,-)_{X,Y}\colon (X,Y) \times (Y,S(X)) \longrightarrow E$$

such that $(f,\,g)_{X,Y} = \phi_{X,Y}^{-1}(g)(f)$. By the naturalness of $\phi_{X,Y}$, we have

(A.1) $(f \circ \theta,\ g)_{X',Y} = (f,\ S(\theta) \circ g)_{X,Y}$

(A.2) $(f,\ g \circ \gamma)_{X,Y'} = (\gamma \circ f,\ g)_{X,Y},$

where $\theta\colon X' \to X$ and $\gamma\colon Y' \to Y$ are arbitrary morphisms. Note that in (A.1), $X, X' \in \mathcal{C}_r$, $f\colon X \to Y$ and $g\colon Y \to S(X')$; in (A.2), $X \in \mathcal{C}_r$, $f\colon X \to Y'$ and $g\colon Y \to S(X)$.

We will call the bilinear form $(-,-)$ defined above the *associated bilinear form* to S. Note that the following direct consequence of the above definition: $F(f) = (f, \phi_{X,Y}(F))_{X,Y}$ for each $F \in D(X,Y)$ and $f \in (X,Y)$.

Let $X, Y \in \mathcal{C}_r$. We claim that for each morphism $f \in (X,Y)$ and $g \in (Y,S(X))$ we have

(A.3) $(f,\ g)_{X,Y} = (g,\ S(f))_{Y,S(X)}.$

In fact, we have

$$\begin{aligned}
(f,\ g)_{X,Y} &= (f \circ \mathrm{Id}_X,\ g)_{X,Y} &&\text{use (A.1)}\\
&= (\mathrm{Id}_X,\ S(f) \circ g)_{X,X} &&\text{use (A.2)}\\
&= (g,\ S(f))_{Y,S(X)}.
\end{aligned}$$

Consider the following composite of R-linear isomorphisms

$$\Phi\colon (X,Y) \xrightarrow{D(\phi_{X,Y})^{-1}} D(Y,S(X)) \xrightarrow{\phi_{Y,S(X)}} (S(X),S(Y)).$$

Note that $D(\phi_{X,Y})^{-1}(f)(g) = (f,g)_{X,Y}$ and $F(g) = (g, \phi_{Y,S(X)}(F))_{Y,S(X)}$ for all $F \in D(Y,S(X))$. Then it is direct to see that $(f,g)_{X,Y} = (g, \Phi(f))_{Y,S(X)}$. Compare this with (A.3), and note that the bilinear form is non-degenerated. We infer that $S(f) = \Phi(f)$. Consequently the functor S is fully faithful.

To see that S is dense, let $X' \in \mathcal{C}_l$. Note that the functor $D(-,X')$ is representable. Assume that $D(-,X') \simeq (X,-)$. Then $D(X,-) \simeq (-,X')$, and this implies that $X \in \mathcal{C}_r$. Therefore we have

$$(-,S(X)) \overset{\phi_X^{-1}}{\simeq} D(X,-) \simeq (-,X').$$

By Yoneda Lemma $S(X) \simeq X'$. Then the functor S is dense, completing the proof. $\square$

Following [26] we will introduce the notion of trace function. For each object $X \in \mathcal{C}_r$, define its *trace function* to be an R-linear map

$$\mathrm{Tr}_X\colon (X,S(X)) \longrightarrow E$$

such that $\mathrm{Tr}_X(f) = (\mathrm{Id}_X, f)_{X,X}$. Then by (A.2), we have $(f,g)_{X,Y} = \mathrm{Tr}_X(g \circ f)$. By (A.3), we get

(A.4) $\mathrm{Tr}_X(g \circ f) = \mathrm{Tr}_Y(S(f) \circ g).$

Let $\mathcal{C}$ be a Hom-finite R-linear category. We call the above obtained sextuple

$$\{S,\ \mathcal{C}_r,\ \mathcal{C}_l,\ \phi,\ (-,-),\ \mathrm{Tr}\}$$

the *generalized Serre duality* on the category $\mathcal{C}$. The functor S is called the *generalized Serre functor* of $\mathcal{C}$, where $\mathcal{C}_r \subseteq \mathcal{C}$ (*resp.* $\mathcal{C}_l \subseteq \mathcal{C}$) is referred as the *domain* (*resp.* the *range*) of the generalized Serre functor.

We say that the category $\mathcal{C}$ *has right Serre duality* ([26]) provided that $\mathcal{C}_r = \mathcal{C}$. In this case, we call S the *right Serre functor* of $\mathcal{C}$. Similarly, we define the notion of *having left Serre duality*. The category $\mathcal{C}$ *has Serre duality* ([7]) provided that it has both right and left Serre duality. This is equivalent to $\mathcal{C}_r = \mathcal{C} = \mathcal{C}_l$.

A.2.   Let $\mathcal{C}$ be an R-linear category which is Hom-finite. Let $\mathcal{C}_r$ and $\mathcal{C}_l$ be the domain and the range of its generalized Serre functor, respectively. In this subsection we will consider the generalized Serre duality on both $\mathcal{C}_r$ and $\mathcal{C}_l$.

Consider the following two conditions on the category $\mathcal{C}$:
 (C) for each $X, X' \in \mathcal{C}_r$ and $Z \in \mathcal{C}$, there exists a morphism $s\colon Z' \to Z$ with $Z' \in \mathcal{C}_r$ such that $(X, Z') \overset{(X,s)}{\simeq} (X, Z)$ and $(Z', X') \overset{(s,X')}{\simeq} (Z, X')$;
 (C') for each $Y, Y' \in \mathcal{C}_l$ and $Z \in \mathcal{C}$, there exists a morphism $s\colon Z \to Z'$ with $Z' \in \mathcal{C}_l$ such that $(Y, Z) \overset{(Y,s)}{\simeq} (Y, Z')$ and $(Z', Y') \overset{(s,Y')}{\simeq} (Z, Y')$.

We have the following result, which is inspired by [14, Theorem 3.4].

Lemma A.2. *Let $\mathcal{C}$ be as above. Then we have*
 (1) *if $\mathcal{C}_l \subseteq \mathcal{C}_r$, then $(\mathcal{C}_r)_r = \mathcal{C}_r$;*
 (2) *conversely, if $(\mathcal{C}_r)_r = \mathcal{C}_r$ and the category $\mathcal{C}$ satisfies the condition (C), then $\mathcal{C}_l \subseteq \mathcal{C}_r$.*

Proof. (1) is obvious from the definition.

To see (2), let $X \in \mathcal{C}_r$. Since $(\mathcal{C}_r)_r = \mathcal{C}_r$, then we have an isomorphism of R-linear functors on $\mathcal{C}_r$: $D(X, -) \simeq (-, X')$, where $X' \in \mathcal{C}_r$. We claim that this isomorphism $D(X, -) \simeq (-, X')$ can be extended on $\mathcal{C}$. Then we get $S(X) = X'$, where S is the generalized Serre functor on $\mathcal{C}$. By Proposition A.1 we have $\mathcal{C}_l = S(\mathcal{C}_r)$. Then we conclude that $\mathcal{C}_l \subseteq \mathcal{C}_r$.

In fact, for each $Z \in \mathcal{C}$, take $s\colon Z' \to Z$ to be the morphism in the condition (C). Then we have a sequence of isomorphisms

$$D(X, Z) \overset{D(X,s)}{\simeq} D(X, Z') \simeq (Z', X') \overset{(s,X')}{\simeq} (Z, X').$$

Note that this composite isomorphism is given by the trace function on X; more precisely, the inverse of this isomorphism is given by $f \longmapsto (g \mapsto \mathrm{Tr}_X(f \circ g))$; here Tr is the trace function given by the generalized Serre duality on $\mathcal{C}_r$. Then we get a natural isomorphism of R-linear functors on $\mathcal{C}$: $D(X, -) \simeq (-, X')$. We are done.   □

Dually, we have the following result.

Lemma A.3. *Let $\mathcal{C}$ be as above. Then we have*
 (1) *if $\mathcal{C}_r \subseteq \mathcal{C}_l$, then $(\mathcal{C}_l)_l = \mathcal{C}_l$;*
 (2) *conversely, if $(\mathcal{C}_l)_l = \mathcal{C}_l$ and the category $\mathcal{C}$ satisfies the condition (C'), then $\mathcal{C}_r \subseteq \mathcal{C}_l$.*   □

The following result seems to be of interest.

Proposition A.4. *Let $\mathcal{C}$ be a Hom-finite R-linear category. Let $\mathcal{C}_r$ and $\mathcal{C}_l$ the domain and the range of the generalized Serre functor on $\mathcal{C}$, respectively. Assume that the category $\mathcal{C}$ satisfies the conditions (C) and (C'). Then the following statements are equivalent:*

(1) *the category $\mathcal{C}_r$ has Serre duality;*
(2) *we have $\mathcal{C}_r = \mathcal{C}_l$;*
(3) *the category $\mathcal{C}_l$ has Serre duality.*

Proof. The implications $(2) \Rightarrow (1)$ and $(1) \Rightarrow (3)$ follow from Lemma A.2(1) and Lemma A.3(1), respectively.

To see $(1) \Rightarrow (2)$, we apply Lemma A.2(2) and then we get $\mathcal{C}_l \subseteq \mathcal{C}_r$. Therefore $(\mathcal{C}_l)_l \supseteq \mathcal{C}_l \cap (\mathcal{C}_r)_l$. Here we use an easy fact that for any full subcategory $\mathcal{D}$ of a category $\mathcal{C}$, one has $\mathcal{D}_l \supseteq \mathcal{C}_l \cap \mathcal{D}$. However by (1), $(\mathcal{C}_r)_l = \mathcal{C}_r$. Hence $(\mathcal{C}_l)_l = \mathcal{C}_l$. Now applying Lemma A.3(2), we get $\mathcal{C}_r \subseteq \mathcal{C}_l$. Then we are done. Similarly we prove the implication $(2) \Rightarrow (3)$. $\qquad\square$

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